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Stability Theorems for Impulsive Functional Differential Equations

Sanjay K. Srivastava
Associate Professor,
Deptt. of Applied Sciences,
Beant College of Engineering and Technology,
Gurdaspur, Punjab

Neeti Bhandari
Research Scholar,
Deptt. of Applied Sciences,
Punjab Technical University,
Jalandhar, Punjab.

Neha Wadhwa
Assistant Professor,
Deptt. of Applied Sciences,
Amritsar College of Engineering and Technology,
Amritsar,

Abstract

In this paper, sufficient conditions are derived for asymptotic stability and uniformly asymptotic stability for impulsive functional differential equation using piecewise continuous differential equation.

Keywords: Stability, Impulsive Functional Differential Equation, Liapunov functional

Introduction

Consider the impulsive functional differential equation

$$\begin{cases} x'(t) = f(t, x_t), & t \neq t_k\ t \geq t_0 \\ \Delta x = I_k(t, (x_t^-)), & t = t_k, k \in Z^+ \end{cases} \quad (1)$$

Where $f: J \times PC \to R^n, \Delta x = x(t) - x(t^-), t_0 < t_1 < \cdots t_k < t_{k+1} < \cdots$, With $t_k \to \infty$ as $k \to \infty$ and $I_k: J \times S(\rho) \to R^n$, where $J = [t_0, \infty)$, $S(\rho) = \{x \in R: |x| < \rho\}. PC = PC([-\tau, 0], R^n)$ denotes the space of piecewise right continuous functions $\varphi: [-\tau, 0] \to R^n$ with sup-norm $\|\varphi\|_\infty = sup_{-\tau \leq s \leq 0}|\varphi(s)|$ and the norm $\|\varphi\|_2 = (\int_{-\tau}^{0} |\varphi(s)|^2 ds)^{1/2,}$ where τ is a positive constant, $\|.\|$ is a norm in $R^n. x_t \in PC$ is defined by $x_t(s) = x(t+s)$ for $-\tau \leq s \leq 0. x'(t)$ denotes the right-hand derivative of $x(t). Z^+$ is the set of all positive integers,

Let $f(t, 0) = 0$ and $J(0) = 0$, then $x(t) = 0$ is the zero solution of (1). Set $PC(\rho) = \{\varphi \in PC: \|\varphi\|_\infty < \rho\}, \forall \rho > 0$.

Definition 1.1

Let σ be the initial time, $\forall\ \sigma \in R$, the zero solution of (1) is said to be

a) stable if , for each $\sigma \geq t_0$ and $\varepsilon > 0$, there is a $\delta = \delta(\sigma, \varepsilon) > 0$ such that , for $\varphi \in PC(\delta)$, a solution $x(t, \sigma, \varphi)$ satisfies $|x(t, \sigma, \varphi)| < \varepsilon$ for $t \geq t_0$.
b) uniformly stable if it is stable and δ in the definition of stability is independent of σ
c) asymptotically stable if it is stable and, for each $t_0 \in R_+$, there is an $\eta = \eta(t_0) > 0$ such that, for $\varphi \in PC(\eta), x(t, \sigma, \varphi) \to 0\ as\ t \to \infty$
d) uniformly asymptotically stable if it is uniformly stable and there is an $\eta > 0$ and , for each $\varepsilon > 0$, a $T = T(\varepsilon) > 0$ such that , for $\varphi \in PC(\eta), |x(t, \sigma, \varphi)| < \varepsilon$ for $t \geq t_0 + T$

Definition 1.2

A functional $V(t, \varphi): J \times PC(\rho) \to R_+$ belong to class $v_o(.)$ (a set of Liapunov like functional) if

a) V is continuous on $[t_{k-1}, t_k) \times PC(\rho)$ for each $k \in Z_+$, and for all $\varphi \in PC(\rho)$ and $k \in Z_+$, the limit $\lim_{(t,\varphi)\to(t_k^-,\varphi)} V(t, \varphi) = V(t_k^-, \varphi)$ exists.
b) V is locally Lipchitzian in φ in each set in $PC(\rho)$ and $V(t, 0) = 0$ The set $\Re$ is defined by $\Re = \{W \in C(R_+, R_+)$: strictly increasing and $W(0) = 0$

Main Results

Theorem 1

Assume that there exist $V_1, V_2 \in v_0(.)$, $W_1, W_2, W_3, W_4 \in \Re$ such that

I. $W_1(|\varphi(0)|) \leq V(t, \varphi) \leq W_2(|\varphi(0)|)$,where $V(t, \varphi) = V_1(t, \varphi) + V_2(t, \varphi)$
II. $V(t_k, x + I_k(t_k, x)) - V(t_k^-, x) \leq 0$
III. $a\, V_1'(t, x_t) + bV_2'(t, x_t) \leq -\lambda(t)\, W_3\, (inf\ \{|x(s)| : t - h \leq s \leq t\})$
IV. $pV_1'(t, x_t) + qV_2'(t, x_t) \leq 0$

where $a^2 + b^2 \neq 0$, $p^2 + q^2 \neq 0\ and\ \int_0^\infty \lambda(s)\, ds = \infty$

(A) Suppose further that there is a $\mu = \mu(\gamma) > 0\ for\ each\ 0 < \gamma < H_1$ such that

$$pV_1'(t, x_t) + qV_2'(t, x_t) \leq -\mu\, V_1'(t, x_t) \quad (2)$$

if $|x(t)| \geq \gamma$. If either $(i)\ a > 0, b > 0\ \ or\ (ii) p \geq 0, q > 0$ hold, then the zero solution of (1) is uniformly and asymptotically stable.

(B) The same is concluded if
$$pV_1'(t,x_t)+qV_2'(t,x_t)\le \mu V_1'(t,x_t)$$
holds in place of (2) and if either $(i)\ a>0,b>0\ or\ (ii) p>0,q>0.$

Proof

We first prove the uniform stability. For given $\varepsilon>0$,we may choose a $\delta=\delta(\varepsilon)>0$ such that $W_2(\delta)<W_1(\varepsilon)$. For any $\sigma\ge t_0$ and $\varphi\in PC_\delta$, let $x(t,\sigma,\varphi)$ be the solution of (1). We will prove that
$$|x(t,\sigma,\varphi)|\le\varepsilon,\qquad t\ge\sigma$$
Let $x(t)=x(t,\sigma,\varphi)$ and $V_1(t)=V_1(t,x_t), V_2(t)=V_2(t,x_t)$ and $V(t)=V(t,x_t)$.
Then by assumption (iv),
$$V'(t,x_t)\le 0,\quad \sigma\le t_{k-1}\le t<t_k,\quad k\in Z^+$$
and so V(t) is non increasing on the interval of the form $[t_{k-1},t_k)$. From condition (ii)
$$V(t_k)-V(t_k^-)=V\big(t_k,x(t_k^-)+I_k(t_k,x(t_k^-))\big)-V(t_k^-,x(t_k^-))\le 0$$
Thus V(t) is non increasing on $[\sigma,\infty)$. We have
$$W_1(|x(t)|)\le V(t)\le V(\sigma)\le W_2(\sigma)<W_1(\varepsilon),\quad t\ge\sigma$$
This implies with the monotonicity of W_1,$|x(t)|<\varepsilon$ for $t\ge\sigma$ and so that the zero solution of (1) is uniformly stable.

To show asymptotic stability, for a given $t_0\in R_+$ and a fixed $0<H_2<H_1$, take $\eta=\eta(t_0)=\delta(t_0,H_2)>0$, where δ is that in the definition of stability and for a given $\varphi\in PC(\eta)$, let $x(t)=x(t,\sigma,\varphi)$ be a solution of (1). Suppose for contradiction that $x(t)\nrightarrow 0$ as $t\to\infty$. Then there is a sequence $\{T_i\}$ and an $\varepsilon_0>0$ with $T_i\to\infty$ and $|x(T_i)|>\varepsilon_0$. Define $\varepsilon_2=W_2^{-1}(\frac{W_1(\varepsilon_0)}{2})$ then there is a sequence $\{s_i\}$ with $s_i\to\infty$ and $|x(s_i)|<\varepsilon_2$. Otherwise there is an $S\ge t_0$ such that $|x(t)|\ge\varepsilon_2$ for $t\ge S$ and
$av_1(t)+bv_2(t)\le$
$av_1(S+h)+bv_2(S+h)-\int_{S+h}^{t}\lambda(s)W_4(\inf\{|x(\sigma)|: s-h\le$ σ≤sds+
S+h≤tk≤t[Vtk−S+h≤tk≤t[Vtk−Vtk−)]

$$\le av_1(S+h)+bv_2(S+h)-W_4(\varepsilon_2)\int_S^t\lambda(s)ds\to-\infty$$
as $t\to\infty$, which contradicts either $av_1(t)+bv_2(t)\ge 0$ if (i) holds or
$$av_1(t)+bv_2(t)\ge -|a|W_2(H_2)-|b|(pv_1(t_0)+qV_2(t_0))/q$$
if (ii) holds.

In Case (A), we may assume $T_{i-1}<s_i<T_i$ by choosing and renumbering if necessary. Then we can take a sequence $\{t_i\}$ such that $s_i<t_i<T_i$, $|x(t_i)|=\varepsilon_2$ and $|x(t)|>\varepsilon_2$ for $t_i<t\le T_i$.
Then $pv_1(T_i)+qv_2(T_i)-\big(pv_1(T_{i-1})+qv_2(T_{i-1})\big)$
$$\le pv_1(T_i)+qv_2(T_i)-\big(pv_1(t_i)+qv_2(t_i)\big)+\sum_{t_i\le t_k\le T_i}[V(t_k)-V(t_k^-)]$$
$$\le -\mu(\varepsilon_2)\big(v_1(T_i)-v_1(t_i)\big)$$
$$\le -\mu(\varepsilon_2)W_1(\varepsilon_0)/2$$
and a contradiction follows from
$$pv_1(T_n)+qv_2(T_n)=pv_1(T_1)+qv_2(T_1)+\sum_{i=2}^{n}\big[pv_1(T_i)+qv_2(T_i)-\big(pv_1(T_{i-1})+qv_2(T_{i-1})\big)\big]+\sum_{T_{i-1}\le t_k\le T_i}[V(t_k)-V(t_k^-)]$$
$$\le pv_1(T_1)+qv_2(T_1)-\frac{(n-1)\mu(\varepsilon_2)W_1(\varepsilon_0)}{2}\to-\infty$$
as $n\to\infty$

In Case (B), we may assume $s_{i-1}<T_i<s_i$ and take $\{t_i\}$ with $T_i<t_i<s_i$, $|x(t_i)|=\varepsilon_2$ and $|x(t)|>\varepsilon_2$ for $T_i\le t<t_i$ so that
$$pv_1(t_i)+qv_2(t_i)-\big(pv_1(t_{i-1})+qv_2(t_{i-1})\big)$$
$$\le pv_1(t_i)+qv_2(t_i)-\big(pv_1(T_i)+qv_2(T_i)\big)+\sum_{T_i\le t_k\le t_i}[V(t_k)-V(t_k^-)]$$
$$\le\mu(\varepsilon_2)(v_1(t_i)-v_1(T_i))$$
$$\le-\mu(\varepsilon_2)W_1(\varepsilon_0)/2$$
This implies a contradiction by the same argument as in case (A)

Therefore, $x(t)\to 0$ as $t\to\infty$. The proof is complete.

Theorem 2.

Assume that there exist $V_1,V_2\in v_0(.)$ and $W_1,W_2,W_3,W_4\in\Re$ such that

a) $W_1|\varphi(0)|\le V(t,\varphi)\le W_2|\varphi(0)|$ where $V(t,\varphi)=V_1(t,\varphi)+V_2(t,\varphi)$
b) $V\big(t_k,x+I_k(t_k,x)\big)-V(t_k^-,x)\le 0,\ k\in Z^+$
c) $aV'_1(t,x_t)+bV'_2(t,x_t)\le -\lambda(t)W_3(\inf\{|x(s)|;t-h\le s\le t\})$ *and* $pV'_1(t,x_t)+qV'_2(t,x_t)\le 0$

Where $a^2+b^2\ne 0, p^2+q^2\ne 0$ *and*
$$\lim_{S\to\infty}\int_t^{t+S}\lambda(s)ds=\infty \text{ uniformly in } t\in R_+$$

A. Suppose that there is a $\mu=\mu(\gamma)>0$ for each $0<\gamma<H_1$ such that
$$pV'_1(t,x_t)+qV'_2(t,x_t)\le-\mu V'_1(t,x_t)\qquad(3)$$
If $|x(t)|\ge\gamma$. If either (i) $a>0,b\ge 0$ or (ii) $p\ge 0,q\ge 0$ hold, then the zero solution of (1) is uniformly asymptotically stable.

B. The same is concluded if (3) is replaced by
$$pV'_1(t,x_t)+qV'_2\le\mu V'_1(t,x_t)$$

And if either (i) $a>0,b\ge 0$ or (ii) $p>0,q\ge 0$ hold

Proof

Uniform Stability can be proven as stability in Theorem 1.

Set $\eta=\delta(H_2)$ for a fixed $0<H_2<H_1$ and δ in the definition of uniform stability. For given $t_0\in R_+,\varphi\in C_\eta$, let $x(t)=x(t,\sigma,\varphi)$ be a solution of (1). Let $\varepsilon>0$ be given and take $\delta=\delta(\varepsilon)>0$ of uniform stability. Define $\delta_1=W_2^{-1}(\frac{W_1(\delta)}{2})$. Choose a $S=S(\varepsilon)>0$ with
$$\int_t^{t+S}\lambda(s)ds>2(|a|W_2(H_2)+|b|W_3(H_2))/W_4(\delta_1)$$
For $t\in R_+$ and an integer $N=N(\varepsilon)\ge 1$ with $N\mu(\delta_1)W_1(\delta)/2>2(|p|W_2(H_2)+|q|W_3(H_2))$

Define $T = T(\varepsilon) = N(S + 2h)$. Suppose, for contradiction, that $||x_t|| \geq \delta$ for $t_0 \leq t \leq t_0 + T$.
In Case (A), for $1 \leq i \leq N$, there is a

$$+(i-1)(S+2h) \leq s_i \leq t_0 + (i-1)(S+2h) + h + S$$

With $|x(s_i)| < \delta_1$.Otherwise $|x(t)| \geq \delta_1$ on this interval and, for $I_i = [t_0 + (i-1)(S+2h) + h, t_0 + (i-1)(S+$ *2h+h+S, v1t=V1(t,xt)* and *v2t=V2(t,xt)*, we have

$$-2(|a|W_2(H_2) + |b|W_3(H_2)) \leq av_1(t_0 + (i-1)(S+2h) + h + S) + bv_2(t_0 + (i-1)(S+2h) + h + S)$$

$$(-av_1(t_0 + (i-1)(S+2h) + h) + bv_2(t_0 + (i-1)(S+2h) + h))$$

$$\leq -\int \lambda(t) W_4(\inf\{|x(s)|: t-h \leq s \leq t\}) ds$$

$$\leq -W_4(\delta_1)\int \lambda(t) \ < \ -2(|a|W_2(H_2) + |b|W_3(H_2))$$

This inequality also holds true as per condition (ii) a contradiction.
From the supposition , for $1 \leq i \leq N$, there is a

$$t_0 + (i-1)(S+2h) + h + S \leq T_i \leq t_0 + i(S+2h)$$

Such that $|x(T_i)| \geq \delta$. Thus, there is an $s_i < t_i < T_i$ with $|x(t_i)| = \delta_1$ and $|x(t)| > \delta_1$ for $t_i < t \leq T_i$. We obtain

$$pv_1\big(t_0 + i(S+2h)\big) + qv_2\big(t_0 + i(S+2h)\big) - \big(pv_1\big(t_0 + (i-1)(S+2h)\big) + qv_2(t_0 + (i-1)(S+2h))\big)$$

$$\leq pv_1(T_i) + qv_2(T_i) - (pv_1(t_i) + qv_2(t_i))$$

$$\leq -\mu(\delta_1)\big(v_1(T_i) - v_1(t_i)\big) \ \leq \ -\mu(\delta_1)W_1(\delta)/2$$

And

$$-2\big(|p|W_2(H_2) + |q|W_3(H_2)\big) \leq \ pv_1\big(t_0 + N(S+2h)\big) + qv_2\big(t_0 + N(S+2h)\big) - (pv_1(t_0) + q(v_2(t_0))$$

$$=\sum_{i=1}^{N}(pv_1\big(t_0 + i(S+2h)\big) + qv_2\big(t_0 + i(S+2h)\big)) -$$

(pv1t0+i−1S+2h+qv2t0+i−1S+2h)

$$\leq -N\mu(\delta_1)W_1(\delta)/2 \ < \ -2(|p|W_2(H_2) + |q|W_3(H_2)),$$

This inequality also holds true as per condition (ii) a contradiction.
In Case (B), we can take, for $1 \leq i \leq N$, $t_0 + (i-1)(2h+S) + h \leq s_i \leq t_0 + i(2h+S)$ with $|x(s_i)| < \delta_1$, $t_0 + (i-1)(2h+S) \leq T_i \leq t_0 + (i-1)(2h+S) + h$ with $|x(T_i)| \geq \delta$ and $T_i < t_i < s_i$ with $|x(t_i)| = \delta_1$, $|x(t)| > \delta_1$ for $T_i \leq t < t_i$ so that

$$pv_1\big(t_0 + i(S+2h)\big) + qv_2\big(t_0 + i(S+2h)\big) - \big(pv_1\big(t_0 + (i-1)(S+2h)\big) + qv_2(t_0 + (i-1)(S+2h))\big)$$

$$\leq pv_1(t_i) + qv_2(t_i) - (pv_1(T_i) + qv_2(T_i))$$

$$\leq \mu(\delta_1)\big(v_1(t_i) - v_1(T_i)\big) \ \leq \ -\mu(\delta_1)W_1(\delta)/2$$

This inequality also holds true as per condition (ii) a contradiction follows from this as in case(A)

Consequently $||x_{t'}|| < \delta$ for some $t_0 \leq t' \leq t_0 + T$ and $|x(t)| < \varepsilon$ for $t \geq t_0 + T$. This completes the proof.

Corollary

If there are $V_1, V_2 \in v_0(.)$ and $W_1, W_2, W_3, W_4 \in \mathfrak{R}$ satisfying

a) $W_1|\varphi(0)| \leq V(t,\varphi) \leq W_2|\varphi(0)|$
b) $0 \leq V(t,\varphi) \leq W_3(||\varphi||)$ *where* $V(t,\varphi) = V_1(t,\varphi) + V_2(t,\varphi)$
c) $V\big(t_k, x + I_k(t_k,x)\big) - V(t_k^-, x) \leq 0$
d) $V'_1(t,x_t) + c_1V'_2(t,x_t) \leq 0$
e) $V'_1(t,x_t) + c_2V'_2(t,x_t) \leq -\lambda(t)W_4(inf\{|x(s)|; t-h \leq s \leq t\})$

Where $c_1 \neq c_2$ either $c_1 \geq 0$ or $c_2 \geq 0$ and $lim_{S\to\infty}\int_t^{t+S}\lambda(s)ds = \infty$ uniformly in $t \in R_+$

Then the zero solution of (1) is uniformly asymptotically stable.

Proof

We may assume that $c_1 > c_2$. Then $c_1 \geq 0$, if $c_2 = 0$

$$V'_1(t,x_t) + c_1V'_2(t,x_t) \leq 0 \leq -V'_1(t,x_t)$$

And the conditions of theorem 2(A ii) are satisfied.
If $c_1 > 0$

$$V'_1(t,x_t) + c_1V'_2(t,x_t) \leq (c_1 - c_2)V'_2(t,x_t) \leq -(\frac{(c_1 - c_2)}{c_1})V'_1(t,x_t)$$

Implies uniform stability by Theorem 2(A ii).

Example Consider the impulsive differential equation

$$x'(t) = -a(t)f\big(x(t)\big) + b(t)g\big(x(t-h)\big)$$

$$x(t_k) - x(t_k^-) = c_kx(t_k^-), \ k \in Z^+$$

Where $a: R_+ \to R_+$, $b: R_+ \to R$, $f, g: R \to R$ are continuous, $xf(x) > 0$, for $x \neq 0, |g(x)| \leq c|f(x)|$ for $c > 0$ and $g(x) \neq 0$ for $x \neq 0$, $|1+c_k| \leq 1, k \in Z^+$ and $\sum_{k=1}^{\infty}[1 - |1 + c_k|] = \infty$
If $\int_t^{t+h}|b(s)|ds$ is bounded, $a(t) - \alpha c|b(t+h)| \geq 0$
For some $\alpha > 1$, and for some $1 \leq \beta \leq \alpha$, $\lambda(t) = a(t) - \beta c|b(t+h)| + (\beta - 1)|b(t)|$ satisfies

$$\lim_{S\to\infty}\int_t^{t+S}\lambda(s)ds = \infty$$

Uniformly in $t \in R_+$, then the zero solution is uniformly asymptotically stable.

Proof

Let $V = V_1 + V_2$ where $V_1(t,\varphi) = |\varphi(0)|$, $V_2(t,\varphi) = \int_{-h}^{0}|b(t+s+h)||g(\varphi(s)|\,ds$
Then $V_2(t,\varphi) \leq \int_t^{t+h}|b(s)|dsW_3(||\varphi||)$ for some function $W_3 \in \mathfrak{R}$
And $V_1(t_k, x + c_kx) - V_1(t_k^-, x) = |(1+c_k)x| - |x| = [1 - |1+c_k|]V(t_k^-, x)$
Let $\lambda_k = 1 - |1 + c_k|$; then $\sum_{k=1}^{\infty}\lambda_k = \infty$. We check that for any $\alpha > 0$, there is a $\beta > 0$ such that $V(t,x_t) \geq \alpha$ implies $V_1(t,x_t) \geq \beta$.
Otherwise we must have $\lim\inf_{t\to\infty}V_1(t,x_t) = 0$
We let $V(t) = V_1(t,x_t) + V_2(t,x_t)$
Then $V(t_k) - V(t_k^-) = V_1\big(t_k, x(t_k^-) + c_kx(t_k^-)\big) - V_1\big(t_k^-, x(t_k^-)\big) \leq 0$

$$V'_1(t,x_t) + \beta V'_2(t,x_t) \leq -(a(t) - \beta c|b(t+h)|)\big|f\big(x(t)\big)\big| - (\beta - 1)|b(t)|\big|g\big(x(t-h)\big)\big| + \sum_{0\leq t_k\leq t}V(t_k) - V(t_k^-)$$

$$\leq -\lambda(t)W_4(\inf\{|x(s)|: t-h \leq s \leq t\})$$

If $||x_t|| \leq H$ for a fixed $0 < H < \infty$ and some function W_4.
If $\beta = 1$, for $\alpha \neq 1$ $V'_1(t,x_t) + \alpha V'_2(t,x_t) \leq 0$
If $\beta > 1$ $V'_1(t,x_t) + 1\,V'_2(t,x_t) \leq 0$
The conditions of the corollary are satisfied and hence the zero solution is uniformly asymptotically stable.

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